Integral input-to-state stabilization in different norms for a class of linear parabolic PDEs

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Abstract—In this paper, we study the problem of integral input-to-state stabilization in different norms for parabolic PDEs with integrable inputs. More precisely, we apply the method of backstepping to design a boundary control law for certain linear parabolic PDEs with destabilizing terms and L^r-inputs, and establish the integral input-to-state stability in the spatial L^p-norm and $W^{1,p}$-norm, respectively, for the closed-loop system, whenever $p \in [1, +\infty]$ and $r \in [p, +\infty]$.

In order to deal with singularities in the case of $p \in [1, 2)$, we employ the approximative Lyapunov method to analyze the stability in different norms. Concerning with the appearance of external inputs, we apply the method of functional analysis and the theory of series to prove the unique existence and regularity of solution to the closed-loop system.

Index Terms—integral input-to-state stability, input-to-state stability, backstepping, approximative Lyapunov method, parabolic equation

I. INTRODUCTION

In the past decade, a great effort has been devoted to developing the theory of input-to-state stability (ISS) for infinite dimensional systems governed by partial differential equations (PDEs); see, e.g., [1], [2]. In particular, the ISS, as well as its variation *integral input-to-state stability* (iISS), of open-loop PDEs, for which the zero solution of zero-input system is asymptotically stable, has been well studied; see, e.g., [1]–[10] and the references therein, while the integral input-to-state stabilization of PDEs with destabilizing terms was less considered, except [6], [11], [12] for the input-to-state stabilization of parabolic PDEs or cascade ODE-PDE systems.

For parabolic PDEs with destabilizing terms, it has been shown that the backstepping is a powerful tool for designing controllers and the classical Lyapunov method (CLM) is well-suited for the stability analysis; see, e.g., [13]. Nevertheless, in the most of reported work, the stability estimates were established in the spatial L^2-norm or H^1-norm; see, e.g., [14]–[23] for the exponential stability of systems without external

inputs, and [11], [12] for the ISS of systems with external inputs, respectively.

However, due to practical applications or from mathematical point of view, stability properties of PDE systems need to be assessed in the sense of various norms; see, e.g., [24]–[26] for the spatial L^1-stability of PDEs without external inputs, and [7], [27], [28] for the ISS and iISS in various norms of parabolic PDEs with external inputs, respectively. Therefore, it is of great interest to study the problem of input-to-state stabilization, or its general framework, i.e., the integral input-to-state stabilization, in different norms for PDEs with destabilizing terms and external inputs.

It is worth noting that, for the (integral) input-to-state stabilization of parabolic PDEs, a boundary controller can be still designed by following the standard procedure of backstepping (see, e.g., [11], [12]), but it is not an easy task to analyze the ISS or iISS (even the exponential stability without involving external inputs) in different norms by using the CLM. For example, consider stability of the following heat equation without external inputs via the CLM:

$$u_t(x,t) = u_{xx}(x,t), (x,t) \in (0,1) \times (0,+\infty),$$
$$u(0,t) = u(1,t) = 0, \quad t \in (0,+\infty).$$

It is well-known that the functional $V(u) := \int_0^1 |u(x,t)|^p dx$ is an appropriate Lyapunov candidate for obtaining the exponential stability in the L^p-norm when $p \geq 2$. However, as indicated in [27], [28], utilization of $V(u)$ may lead to singularities in the case of $p \in [1,2)$. Indeed,

- for $p = 1$, the functional $\frac{d}{dt} \int_0^1 |u| dx$ is singular due to the fact that $g(s) := |s|$ is not differentiable at $s = 0$;
- for $1 < p < 2$, it holds (formally) that

$$\frac{d}{dt} \int_0^1 |u|^p dx = p \int_0^1 |u|^{p-1} \mathrm{sgn}(u) u_t dx$$

$$= p \int_0^1 |u|^{p-1} \mathrm{sgn}(u) u_{xx} dx$$

$$= -p \int_0^1 \frac{d}{du}(|u|^{p-1}\mathrm{sgn}(u)) u_x^2 dx,$$

with $\frac{\mathrm{d}}{\mathrm{d}s}(|s|^{p-1}\mathrm{sgn}(s))$ being singular at $s = 0$, where $\mathrm{sgn}(\cdot)$ denotes the standard sign function.

In order to overcome the aforementioned obstacle in the application of Lyapunov arguments, the authors of [27], [28] introduced the approximative Lyapunov method (ALM) and established for a class of nonlinear open-loop parabolic systems the iISS in the L^1-norm under various boundary conditions [27], and the iISS in the L^p-norm and $W^{1,p}$-norm with any $p \in [1, 2)$ under Dirichlet boundary conditions [28], respectively. Nevertheless, the integral input-to-state stabilization in different norms for parabolic PDEs is less studied.

In this paper, we consider the problem of integral input-to-state stabilization in the spatial L^p-norm and $W^{1,p}$-norm for certain linear parabolic PDEs with external inputs, whenever $p \in [1, +\infty]$. We follow the standard procedure of backstepping to design a boundary controller, while employ the ALM to assess the iISS in different norms for the closed-loop system.

It is worth noting that, due to the appearance of external inputs, the invertibility of integral transformation cannot be proved via deriving the kernel functions, but it is proved by using the method of functional analysis. In addition, since the ISS or iISS requires that the norm of the solutions should be consistent with the norm of initial data, the stability in the L^p-norm or $W^{1,p}$-norm with $p \in [1, 2)$ cannot be obtained via Sobolev embedding theorems in the Lyapunov arguments.

This paper is organized as follows. Section II introduces the problem setting, main result and technical line. Section III and Section IV present the iISS in different norms for the target system and the closed-loop system, respectively. Concluding remarks are given in Section V.

Notation Let $\mathbb{N}_0$, $\mathbb{N}_+$, $\mathbb{R}$, $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ be the set of nonnegative integers, positive integers, real numbers, positive real numbers and nonnegative real numbers, respectively.

For $q \in [1, +\infty]$ and an arbitrary open or closed domain Ω in $\mathbb{R}^1$ or $\mathbb{R}^2$, the notations of $L^q(\Omega)$ and $W^{1,q}(\Omega)$ denote the standard Lebesgue spaces and Sobolev spaces with elements defined over Ω and equipped with the norm $\|v\|_{L^q(\Omega)}$ and $\|v\|_{W^{1,q}(\Omega)}$, respectively; see, e.g., [29]. Let $C(\Omega) := \{v : \Omega \to \mathbb{R} | \ v$ is continuous on $\Omega\}$. For $i \in \mathbb{N}_+$, $C^i(\Omega) := \{v : \Omega \to \mathbb{R}| \ v$ has continuous derivatives up to order i on $\Omega\}$ equipped with the norm $\|v\|_{C^i(\Omega)} := \sum_{|j| \leq i} \sup_\Omega |D^j v|$.

Let $Q_\infty := (0, 1) \times \mathbb{R}_{>0}$ and $\overline{Q}_\infty := [0, 1] \times \mathbb{R}_{\geq 0}$. For $T \in \mathbb{R}_{>0}$, let $Q_T := (0, 1) \times (0, T)$ and $\overline{Q}_T := [0, 1] \times [0, T]$. Let $C^{2,1}(\overline{Q}_T) := \{v : \overline{Q}_T \to \mathbb{R}| \ v, v_x, v_{xx}, v_t \in C(\overline{Q}_T)\}$. For $\sigma \in \mathbb{R}_{>0}$, the notations of $\mathcal{H}^\sigma([0,1])$ and $\mathcal{H}^{\sigma,\frac{\sigma}{2}}(\overline{Q}_T)$ denote Hölder spaces that are defined in [30, Chap. I].

For a function $v : \overline{Q}_T \to \mathbb{R}$, the notation $v[t]$ denotes the profile at $t \in [0, T]$, i.e., $v[t](y) = v(y, t)$ for all $y \in [0, 1]$.

Let $\mathcal{K} := \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}| \ \gamma(0) = 0, \gamma$ is continuous, strictly increasing$\}$, $\mathcal{L} := \{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}| \ \lim_{s \to \infty} \gamma(s) = 0, \gamma$ is continuous, strictly increasing$\}$, $\mathcal{KL} := \{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}| \ \beta(\cdot, t) \in \mathcal{K}, \forall t \in \mathbb{R}_{\geq 0}$ and $\beta(s, \cdot) \in \mathcal{L}, \forall s \in \mathbb{R}_{\geq 0}\}$.

Throughout this paper, let

$$D := \{(x, y) \in \mathbb{R}^2| \ 0 \leq y \leq x \leq 1\}.$$

For $k \in C^2(D)$ that satisfies (3) (see [21]), denote

$$\alpha_1 := \max_{(x,y) \in D} |k(x, y)|, \quad \alpha_2 := \max_{0 \leq x \leq 1} |k(x, x)|,$$

$$\alpha_3 := \max_{(x,y) \in D} |k_x(x, y)|, \quad \alpha_4 := \max_{(x,y) \in D} |k_{xx}(x, y)|,$$

$$\alpha_5 := \max_{0 \leq x \leq 1} \left| \frac{\mathrm{d}}{\mathrm{d}x}(k(x, x)) + (k_x(x, y))|_{y=x} \right|.$$

II. PROBLEM SETTING, MAIN RESULT AND TECHNICAL LINE

A. Problem Setting and Main Result

In this paper, we consider the following 1-D linear parabolic equation with mixed boundary conditions:

$$w_t(x, t) = w_{xx}(x, t) + c(x)w(x, t)$$
$$+ f(x, t), (x, t) \in Q_\infty, \quad (1\mathrm{a})$$
$$w(0, t) = 0, t \in \mathbb{R}_{>0}, \quad (1\mathrm{b})$$
$$w_x(1, t) = U(t), t \in \mathbb{R}_{>0}, \quad (1\mathrm{c})$$
$$w(x, 0) = w_0(x), x \in (0, 1), \quad (1\mathrm{d})$$

where c, f, w_0 and U are given functions, among which f represents distributed in-domain disturbances and U represents boundary control inputs. It is well-known that the open zero-input system (i.e., $f \equiv 0$ and $U \equiv 0$) is unstable for large c; see, e.g., [13].

The objective of this work is to stabilize system (1) in the framework of iISS theory and to establish stability estimates in different norms under the boundary feedback control law

$$U(t) := -k(1, 1)w(1, t) - \int_0^1 k_x(1, y)w(y, t)\mathrm{d}y, \quad (2)$$

where k is a kernel function that is defined over D and satisfies

$$k_{xx}(x, y) - k_{yy}(x, y) = (\lambda + c(y))k(x, y), \quad (3\mathrm{a})$$

$$2\frac{\mathrm{d}}{\mathrm{d}x}(k(x, x)) = \lambda + c(x), \quad (3\mathrm{b})$$

$$k(x, 0) = 0, \quad (3\mathrm{c})$$

with λ being an arbitrarily positive constant.

Throughout this paper, for well-posedness and stability analysis, we assume that

$$c \in C([0, 1]),$$

and choose

$$\mathcal{U} := \mathcal{H}^{\theta, \frac{\theta}{2}}(\overline{Q}_\infty)$$

as an admissible set of in-domain disturbances with some $\theta \in (0, 1)$, i.e.,

$$f \in \mathcal{U}. \quad (4)$$

Let $w_0 \in \mathcal{H}^{2+\theta}([0,1])$ satisfy the compatible conditions

$$w_0(0) = 0, \tag{5a}$$

$$w_{0x}(1) = -k(1,1)w_0(1) - \int_0^1 k_x(1,y)w_0(y)\mathrm{d}y. \tag{5b}$$

We consider

$$p \in [1, +\infty] \quad \text{and} \quad r \in [p, +\infty]$$

that are given arbitrarily.

Definition 1: System (1) is said to be L^r-input-to-state stable (L^r-ISS) in the spatial L^p-norm, or $W^{1,p}$-norm, with respect to (w.r.t.) the in-domain disturbance $f \in \mathcal{U}$, if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the solution to (1) satisfies for all $t \in \mathbb{R}_{>0}$:

$$\|w(\cdot,t)\|_{L^p(0,1)} \leq \beta \left(\|w_0\|_{L^p(0,1)}, t \right) + \gamma \left(\|f\|_{L^r((0,t);L^p(0,1))} \right), \tag{6}$$

or

$$\|w(\cdot,t)\|_{W^{1,p}(0,1)} \leq \beta \left(\|w_0\|_{W^{1,p}(0,1)}, t \right) + \gamma \left(\|f\|_{L^r((0,t);W^{1,p}(0,1))} \right). \tag{7}$$

In particular, system (1) is said to be input-to-state stable (ISS) in the spatial L^p-norm, or $W^{1,p}$-norm, w.r.t. the in-domain disturbance $f \in \mathcal{U}$, if (6), or (7), is satisfied with $r = +\infty$.

The following theorem is the main result obtained in this paper.

Theorem 1: Under the boundary feedback control law (2), system (1) admits a unique classical solution w that belongs to $C^{2,1}(\overline{Q}_T)$. Moreover, system (1) is L^r-ISS both in the spatial L^p-norm and in the spatial $W^{1,p}$-norm.

B. Technical Line

The proof of Theorem 1 is divided into three steps, among which Step 2 and Step 3 are presented in Section III and Section IV, respectively.

Step 1. By virtue of the standard procedure of designing a boundary backstepping control for parabolic PDEs (see, e.g., [1], [11], [12], [19]), we use the integral transformation

$$u(x,t) := w(x,t) + \int_0^x k(x,y)w(y,t)\mathrm{d}y, \tag{8}$$

to transform system (1) into the following target system

$$u_t(x,t) = u_{xx}(x,t) - \lambda u(x,t) + f(x,t)$$
$$+ \int_0^x k(x,y)f(y,t)\mathrm{d}y, (x,t) \in Q_\infty, \tag{9a}$$

$$u(0,t) = u_x(1,t) = 0, t \in \mathbb{R}_{>0}, \tag{9b}$$

$$u(x,0) = u_0(x), x \in (0,1), \tag{9c}$$

where $u_0(x) := w_0(x) + \int_0^x k(x,y)w_0(y)\mathrm{d}y$. The proof is standard and thus is omitted in this paper.

It is worth noting that the existence and uniqueness of $k \in C^2(D)$ have been proven by using the method of successive approximations in, e.g., [21].

Step 2. We prove the L^r-ISS of system (9) in the spatial L^p-norm and $W^{1,p}$-norm, respectively, by using the ALM.

The main idea of ALM is to estimate

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \rho_\tau^p(u)\mathrm{d}x \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(\rho_\tau^p(u) + \rho_\tau^p(u_x) \right) \mathrm{d}x$$

other than

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 |u|^p\mathrm{d}x \quad \text{or} \quad \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left(|u|^p + |u_x|^p \right) \mathrm{d}x,$$

so as to avoid singularities in the classical Lyapunov arguments when $p \in [1,2)$, where for any fixed $\tau \in \mathbb{R}_{>0}$ the function $\rho_\tau(\cdot)$ is defined by

$$\rho_\tau(s) := \begin{cases} |s|, & |s| \geq \tau, \\ -\dfrac{s^4}{8\tau^3} + \dfrac{3s^2}{4\tau} + \dfrac{3\tau}{8}, & |s| < \tau. \end{cases} \tag{10}$$

It is clear that $\rho_\tau \in C^2(\mathbb{R})$ and satisfies for any $s \in \mathbb{R}$:

$$\rho_\tau'(0) = 0, \ 0 \leq |s| \leq \rho_\tau(s), \ |\rho_\tau'(s)| \leq 1, \tag{11a}$$

$$0 \leq \rho_\tau''(s) = \begin{cases} 0, & |s| \geq \tau, \\ \dfrac{3}{2\tau}\left(1 - \dfrac{s^2}{\tau^2}\right), & |s| < \tau, \end{cases} \tag{11b}$$

$$0 \leq \rho_\tau(s) - \frac{3\tau}{8} \leq \rho_\tau'(s)s \leq \rho_\tau(s) \leq |s| + \frac{3\tau}{8}. \tag{11c}$$

Moreover, the second inequality of (11a) and the fourth inequality of (11c) guarantee that

$$\lim_{\tau \to 0^+} \int_0^1 \rho_\tau^p(u)\mathrm{d}x = \int_0^1 |u|^p\mathrm{d}x,$$

$$\lim_{\tau \to 0^+} \int_0^1 \left(\rho_\tau^p(u) + \rho_\tau^p(u_x) \right) \mathrm{d}x = \int_0^1 \left(|u|^p + |u_x|^p \right) \mathrm{d}x.$$

Step 3. By using the method of functional analysis (see, e.g., [19]) and the theory of series, we establish estimates in different norms for operators related to the inverse transformation of (8). Then Theorem 1 follows from the estimates of u, k, and the invertibility of (8).

Note that, due to the appearance of external disturbances, it is hard to find an inverse transformation of (8) that has a form

$$w(x,t) := u(x,t) + \int_0^x l(x,y)u(y,t)\mathrm{d}y$$

with some kernel function l.

III. THE L^r-ISS OF THE TARGET SYSTEM

We first present a well-posedness result for the target system (9).

Lemma 1: The target system (9) admits a unique classical solution u that belongs to $C^{2,1}(\overline{Q}_T)$ and has the derivative u_{xt} a.e. in Q_T with $u_{xt} \in L^2(Q_T)$ for any $T \in \mathbb{R}_{>0}$.

Proof: In view of $k \in C^2(D)$ (see [21]) and the condition (5) with $w_0 \in \mathcal{H}^{2+\theta}([0,1])$, it is clear that $u_0 \in \mathcal{H}^{2+\theta}([0,1])$ and the compatibility conditions $u_0(0) = u_{0x}(1) = 0$ hold true. In addition, since $f \in \mathcal{U}$ and $k \in C^2(D)$, it follows that $f(x,t) + \int_0^x k(x,y)f(y,t)\mathrm{d}y \in \mathcal{H}^{\theta,\frac{\theta}{2}}(\overline{Q}_\infty)$.

The theory of linear parabolic PDEs (see, e.g., [30, Chap. IV, Theorem 5.3]) guarantees that the target system (9) admits

a unique classical solution $u \in \mathcal{H}^{2+\theta,1+\frac{\theta}{2}}(\overline{Q}_T)$ for any $T \in \mathbb{R}_{>0}$. Note that $\mathcal{H}^{2+\theta,1+\frac{\theta}{2}}(\overline{Q}_T) \hookrightarrow C^{2,1}(\overline{Q}_T)$. Thus, the solution belongs to $C^{2,1}(\overline{Q}_T)$. In addition, in view of the proof of [30, Chap. V, Lemma 7.2], u has the derivative u_{xt} a.e. in Q_T and $u_{xt} \in L^2(Q_T)$. ∎

The following result presents the L^r-ISS of the target system (9).

Proposition 1: The target system (9) is L^r-ISS in the spatial L^p-norm and $W^{1,p}$-norm, having the estimate for all $t \in \mathbb{R}_{>0}$:

$$\|u(\cdot,t)\|_{L^p(0,1)} \leq e^{-\underline{\lambda}t}\|u_0\|_{L^p(0,1)} \\ + (1+\alpha_1)R\|f\|_{L^r((0,t);L^p(0,1))}, \quad (12)$$

and

$$\|u(\cdot,t)\|_{W^{1,p}(0,1)} \leq e^{-\underline{\lambda}t}\|u_0\|_{W^{1,p}(0,1)} \\ + \widetilde{R}\|f\|_{L^r((0,t);W^{1,p}(0,1))}, \quad (13)$$

respectively, where $\underline{\lambda} := \lambda$ for $p = 1$ or $\underline{\lambda} := \lambda - \varepsilon$ for $p \in (1,+\infty]$, $\varepsilon \in (0,\lambda)$ is an arbitrarily positive constant,

$$R := \begin{cases} \left(\dfrac{r-1}{\underline{\lambda}r}\right)^{\frac{r-1}{r}}, & p = 1, \\ \left(\dfrac{\varepsilon p}{p-1}\right)^{\frac{1-p}{p}} \left(\dfrac{r-p}{rp\underline{\lambda}}\right)^{\frac{r-p}{rp}}, & p \in (1,+\infty), \\ \dfrac{1}{\varepsilon}, & p = +\infty, \end{cases}$$

and $\widetilde{R} := (1+\alpha_1+\alpha_2)R$.

Proof: In the sequel, let $\rho_\tau(s)$ be defined by (10), and $F(x,t) := f(x,t) + \int_0^x k(x,y)f(y,t)\mathrm{d}y$. We proceed to the proof in 2 steps.

Step 1. We prove (12). For $p \in (1,+\infty)$, by integrating by parts, we have

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_\tau^p(u)\mathrm{d}x$$
$$= \int_0^1 \rho_\tau^{p-1}(u)\rho_\tau'(u)u_t\mathrm{d}x$$
$$= \int_0^1 \rho_\tau^{p-1}(u)\rho_\tau'(u)(u_{xx} - \lambda u + F)\mathrm{d}x$$
$$= -\int_0^1 \left((p-1)\rho_\tau^{p-2}(u)(\rho_\tau'(u))^2 + \rho_\tau^{p-1}(u)\rho_\tau''(u)\right)u_x^2\mathrm{d}x$$
$$\quad -\int_0^1 \rho_\tau^{p-1}(u)\rho_\tau'(u)\lambda u\mathrm{d}x + \int_0^1 \rho_\tau^{p-1}(u)\rho_\tau'(u)F\mathrm{d}x. \quad (14)$$

We infer from (11a) and (11b) that for all $s \in \mathbb{R}$:

$$\phi(s) := (p-1)\rho_\tau^{p-2}(s)(\rho_\tau'(s))^2 + \rho_\tau^{p-1}(s)\rho_\tau''(s) \geq 0. \quad (15)$$

By (11a) and (11c), we deduce that

$$-\int_0^1 \rho_\tau^{p-1}(u)\rho_\tau'(u)\lambda u\mathrm{d}x$$
$$\leq -\int_0^1 \lambda\rho_\tau^{p-1}(u)\left(\rho_\tau(u) - \frac{3\tau}{8}\right)\mathrm{d}x$$
$$= -\int_0^1 \lambda\rho_\tau^p(u)\mathrm{d}x + \frac{3\tau}{8}\int_0^1 \lambda\rho_\tau^{p-1}(u)\mathrm{d}x. \quad (16)$$

By (11a) and the Young's inequality with $\varepsilon \in \mathbb{R}_{>0}$, we have

$$\int_0^1 \rho_\tau^{p-1}(u)\rho_\tau'(u)F\mathrm{d}x$$
$$\leq \int_0^1 \rho_\tau^{p-1}(u)|F|\mathrm{d}x$$
$$\leq \varepsilon\int_0^1 \rho^p(u)\mathrm{d}x + C(\varepsilon,p)\int_0^1 |F|^p\mathrm{d}x, \quad (17)$$

where $C(\varepsilon,p) := \frac{1}{p}\left(\frac{\varepsilon p}{p-1}\right)^{1-p}$, and $\varepsilon \in \mathbb{R}_{>0}$ will be determined later.

By (14), (15), (16) and (17), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_\tau^p(u)\mathrm{d}x$$
$$\leq -p(\lambda-\varepsilon)\int_0^1 \rho_\tau^p(u)\mathrm{d}x + pC(\varepsilon,p)\int_0^1 |F|^p\mathrm{d}x$$
$$\quad + \frac{3}{8}\tau p\int_0^1 \lambda\rho_\tau^{p-1}(u)\mathrm{d}x. \quad (18)$$

Choosing $\varepsilon \in (0,\lambda)$ and applying the Gronwall's inequality, it follows that

$$\int_0^1 \rho_\tau^p(u(x,t))\mathrm{d}x$$
$$\leq e^{-p(\lambda-\varepsilon)t}\int_0^1 \rho_\tau^p(u_0(x))\mathrm{d}x$$
$$\quad + pC(\varepsilon,p)\int_0^t e^{-p(\lambda-\varepsilon)(t-s)}\int_0^1 |F(x,s)|^p\mathrm{d}x\mathrm{d}s$$
$$\quad + \tau\int_0^t e^{-p(\lambda-\varepsilon)(t-s)}\psi_1(s)\mathrm{d}s, \forall t \in \mathbb{R}_{>0}, \quad (19)$$

where $\psi_1(s) := \frac{3}{8}p\int_0^1 \lambda\rho_\tau^{p-1}(u(x,s))\mathrm{d}x$.

Letting $\tau \to 0^+$ and using the Hölder's inequality, we obtain

$$\|u(\cdot,t)\|_{L^p(0,1)}^p$$
$$\leq e^{-p(\lambda-\varepsilon)t}\|u_0\|_{L^p(0,1)}^p + pC(\varepsilon,p)\int_0^t e^{-p(\lambda-\varepsilon)(t-s)}\overline{F}(s)\mathrm{d}s$$
$$\leq e^{-p(\lambda-\varepsilon)t}\|u_0\|_{L^p(0,1)}^p$$
$$\quad + pC(\varepsilon,p)\|\overline{F}\|_{L^{r_1}(0,t)}\|e^{-p(\lambda-\varepsilon)(t-\cdot)}\|_{L^{r_2}(0,t)}$$
$$\leq e^{-p(\lambda-\varepsilon)t}\|u_0\|_{L^p(0,1)}^p + R^p\|\overline{F}\|_{L^{r_1}(0,t)}, \forall t \in \mathbb{R}_{>0}, \quad (20)$$

where $\overline{F}(s) := \int_0^1 |F(x,s)|^p\mathrm{d}x$, $R := (pC(\varepsilon,p))^{\frac{1}{p}}\left(\frac{1}{p(\lambda-\varepsilon)r_2}\right)^{\frac{1}{r_2 p}}$, $r_1 := \frac{r}{p}$ and $r_2 \in [1,+\infty]$ satisfying $\frac{1}{r_1} + \frac{1}{r_2} = 1$.

Note that

$$\|\overline{F}\|_{L^{r_1}(0,t)} = \left(\int_0^t \left(\int_0^1 |F(x,s)|^p\mathrm{d}x\right)^{\frac{1}{p}pr_1}\mathrm{d}s\right)^{\frac{1}{pr_1}p}$$
$$= \|F\|_{L^r((0,t);L^p(0,1))}^p. \quad (21)$$

It follows from (20), (21) and the equality

$$(a+b)^\varsigma \leq a^\varsigma + b^\varsigma, \forall a,b \in \mathbb{R}_{\geq 0}, \forall \varsigma \in [0,1]$$

that

$$\|u(\cdot,t)\|_{L^p(0,1)} \leq e^{-(\lambda-\varepsilon)t}\|u_0\|_{L^p(0,1)} + R\|F\|_{L^r((0,t);L^p(0,1))}. \quad (22)$$

In the case of $p = 1$, it suffices to note that (17) becomes

$$\int_0^1 \rho_\tau^{p-1}(u)\rho_\tau'(u)F\mathrm{d}x = \int_0^1 \rho_\tau'(u)F\mathrm{d}x \leq \int_0^1 |F|\mathrm{d}x.$$

Then (18) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_\tau(u)\mathrm{d}x \leq -\lambda\int_0^1 \rho_\tau(u)\mathrm{d}x + \int_0^1 |F|\mathrm{d}x + \frac{3}{8}\tau\lambda,$$

which implies that

$$\|u(\cdot,t)\|_{L^1(0,1)} \leq e^{-\lambda t}\|u_0\|_{L^1(0,1)} + R\|F\|_{L^r((0,t);L^1(0,1))}, \quad (23)$$

with $R := \left(\frac{r-1}{\lambda r}\right)^{\frac{r-1}{r}}$.

For the ISS in the spatial L^∞-norm of system (9), it suffices to note that (20) becomes

$$\|u(\cdot,t)\|_{L^p(0,1)}^p \leq e^{-p(\lambda-\varepsilon)t}\|u_0\|_{L^p(0,1)}^p$$
$$+ pC(\varepsilon,p)\int_0^t e^{-p(\lambda-\varepsilon)(t-s)}\|\overline{F}\|_{L^\infty(0,t)}\mathrm{d}s$$
$$\leq e^{-p(\lambda-\varepsilon)t}\|u_0\|_{L^p(0,1)}^p$$
$$+ \frac{C(\varepsilon,p)}{\lambda-\varepsilon}\|F\|_{L^\infty((0,t);L^p(0,1))}^p,$$

which leads to

$$\|u(\cdot,t)\|_{L^p(0,1)} \leq e^{-(\lambda-\varepsilon)t}\|u_0\|_{L^p(0,1)}$$
$$+ \left(\frac{C(\varepsilon,p)}{\lambda-\varepsilon}\right)^{\frac{1}{p}}\|F\|_{L^\infty((0,t);L^p(0,1))}.$$

Letting $p \to +\infty$, we obtain

$$\|u(\cdot,t)\|_{L^\infty(0,1)} \leq e^{-(\lambda-\varepsilon)t}\|u_0\|_{L^\infty(0,1)}$$
$$+ \frac{1}{\varepsilon}\|F\|_{L^\infty((0,t);L^\infty(0,1))}. \quad (24)$$

In view of (22), (23) and (24), and noting that

$$\|F\|_{L^r((0,t);L^p(0,1))} \leq (1+\alpha_1)\|f\|_{L^r((0,t);L^p(0,1))},$$

we conclude that system (9) is L^r-ISS in the spatial L^p-norm having the estimate (12), whenever $p \in [1,+\infty]$ and $r \in [p,+\infty]$.

Step 2. We prove (13). Recalling the regularity of u (see Lemma 1), we calculate in the case of $p \in (1,+\infty)$:

$$\frac{1}{p}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_\tau^p(u_x)\mathrm{d}x$$
$$= \int_0^1 \rho_\tau^{p-1}(u_x)\rho_\tau'(u_x)u_{xt}\mathrm{d}x$$
$$= \left(\rho_\tau^{p-1}(u_x)\rho_\tau'(u_x)u_t\right)|_{x=0}^{x=1}$$
$$- \int_0^1 \left(\rho_\tau^{p-1}(u_x)\rho_\tau'(u_x)\right)_x u_t\mathrm{d}x$$

$$= -\int_0^1 \left(\rho_\tau^{p-1}(u_x)\rho_\tau'(u_x)\right)_x (u_{xx} - \lambda u + F)\,\mathrm{d}x$$
$$= -\int_0^1 \phi(u_x)u_{xx}^2\mathrm{d}x + \int_0^1 \left(\rho_\tau^{p-1}(u_x)\rho_\tau'(u_x)\right)_x \lambda u\mathrm{d}x$$
$$- \int_0^1 \left(\rho_\tau^{p-1}(u_x)\rho_\tau'(u_x)\right)_x F\mathrm{d}x$$
$$= -\int_0^1 \phi(u_x)u_{xx}^2\mathrm{d}x - \int_0^1 \rho_\tau^{p-1}(u_x)\rho_\tau'(u_x)\lambda u_x\mathrm{d}x$$
$$+ \int_0^1 \rho_\tau^{p-1}(u_x)\rho_\tau'(u_x)F_x\mathrm{d}x \quad \text{for a.e. } t \in \mathbb{R}_{>0}, \quad (25)$$

where ϕ and F are the same as defined earlier.

Proceeding in the same way as (18), we deduce by (25) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_\tau^p(u_x)\mathrm{d}x$$
$$\leq -p(\lambda-\varepsilon)\int_0^1 \rho_\tau^p(u_x)\mathrm{d}x + pC(\varepsilon,p)\int_0^1 |F_x|^p\mathrm{d}x$$
$$+ \frac{3}{8}\tau p\int_0^1 \lambda\rho_\tau^{p-1}(u_x)\mathrm{d}x \quad \text{for a.e. } t \in \mathbb{R}_{>0}, \quad (26)$$

where $\varepsilon \in (0,\lambda)$ is an arbitrary constant.

Combining (18) with (26), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\rho_\tau^p(u) + \rho_\tau^p(u_x)\right)\mathrm{d}x$$
$$\leq -p(\lambda-\varepsilon)\int_0^1 \left(\rho_\tau^p(u) + \rho_\tau^p(u_x)\right)\mathrm{d}x$$
$$+ pC(\varepsilon,p)\int_0^1 \left(|F|^p + |F_x|^p\right)\mathrm{d}x$$
$$+ \frac{3}{8}\tau p\int_0^1 \lambda(\rho_\tau^{p-1}(u) + \rho_\tau^{p-1}(u_x))\mathrm{d}x \quad \text{for a.e. } t \in \mathbb{R}_{>0}.$$

Applying the Gronwall's inequality, we have

$$\int_0^1 \left(\rho_\tau^p(u(x,t)) + \rho_\tau^p(u_x(x,t))\right)\mathrm{d}x$$
$$\leq e^{-p(\lambda-\varepsilon)t}\int_0^1 \left(\rho_\tau^p(u_0(x)) + \rho_\tau^p(u_{0x}(x))\right)\mathrm{d}x$$
$$+ pC(\varepsilon,p)\int_0^t e^{-p(\lambda-\varepsilon)(t-s)}\int_0^1 \left(|F|^p + |F_x|^p\right)\mathrm{d}x\mathrm{d}s$$
$$+ \tau\int_0^t e^{-p(\lambda-\varepsilon)(t-s)}\psi_2(s)\mathrm{d}s, \forall t \in \mathbb{R}_{>0},$$

where $\psi_2(s) := \psi_1(s) + \frac{3}{8}p\int_0^1 \lambda\rho_\tau^{p-1}(u_x(x,s))\mathrm{d}x$.

Letting $\tau \to 0^+$ and using the Hölder's inequality, we have for all $t \in \mathbb{R}_{>0}$:

$$\|u(\cdot,t)\|_{W^{1,p}(0,1)}^p$$
$$\leq e^{-p(\lambda-\varepsilon)t}\|u_0\|_{W^{1,p}(0,1)}^p$$
$$+ pC(\varepsilon,p)\int_0^t e^{-p(\lambda-\varepsilon)(t-s)}\widetilde{F}(s)\mathrm{d}s$$
$$\leq e^{-p(\lambda-\varepsilon)t}\|u_0\|_{W^{1,p}(0,1)}^p$$
$$+ pC(\varepsilon,p)\|\widetilde{F}\|_{L^{r_1}(0,T)}\|e^{-p(\lambda-\varepsilon)(T-\cdot)}\|_{L^{r_2}(0,T)}$$

$$\leq e^{-p(\lambda-\varepsilon)t}\|u_0\|_{W^{1,p}(0,1)}^p + R^p\|\widetilde{F}\|_{L^{r_1}(0,t)}, \qquad (27)$$

where $\widetilde{F}(s) := \int_0^1 (|F(x,s)|^p + |F_x(x,s)|^p)\mathrm{d}x$, and r_1, r_2, R are the same as in (20).

Note that

$$\|\widetilde{F}\|_{L^{r_1}(0,t)} = \|F\|_{L^r((0,t);W^{1,p}(0,1))}^p$$
$$\leq (1 + \alpha_1 + \alpha_2)^p \|f\|_{L^r((0,t);W^{1,p}(0,1))}^p.$$

We infer from (27) that the estimate (13) holds true.

The case of $p = 1$ and $p = +\infty$ can be argued in the same way as (23) and (24), respectively. The proof is complete. ∎

IV. THE L^r-ISS OF THE ORIGINAL SYSTEM

In this section, we prove the L^r-ISS in different norms for the original system (1) under the boundary control law (2).

We first prove the following two lemmas that are needed for the well-posedness and stability analysis of the closed-loop system.

Lemma 2: Let $k \in C^2(D)$ be the solution to (3). Let $\mathscr{D}$ be either $L^p(0,1)$ or $W^{1,p}(0,1)$. For any $t \in \mathbb{R}_{>0}$, define the linear operator $K : \mathscr{D} \to \mathscr{D}$ by

$$u[t](x) := (Kw)[t](x)$$
$$:= w[t](x) + \int_0^x k(x,y)w[t](y)\mathrm{d}y. \qquad (28)$$

Then K has a uniformly (with respect to t) bounded linear inverse $K^{-1}: \mathscr{D} \to \mathscr{D}$.

Lemma 3: Let $k \in C^2(D)$ be the solution to (3). For $T \in \mathbb{R}_{>0}$, let $\mathcal{D} := C^{2,1}(\overline{Q}_T)$. Define the linear operator $K : \mathcal{D} \to \mathcal{D}$ by

$$u(x,t) := (Kw)(x,t)$$
$$:= w(x,t) + \int_0^x k(x,y)w(y,t)\mathrm{d}y.$$

Then K has a bounded linear inverse $K^{-1}: \mathcal{D} \to \mathcal{D}$.

Proof of Lemma 2: We proceed to the proof in 2 steps.

Step 1. For $\mathscr{D} = L^p(0,1)$, we prove that K has a uniformly bounded linear inverse $K^{-1}: L^p(0,1) \to L^p(0,1)$.

For any fixed $t \in \mathbb{R}_{>0}$, let

$$v[t](x) := \int_0^x k(x,y)w[t](y)\mathrm{d}y.$$

Then (28) is equivalent to

$$w[t](x) = u[t](x) - v[t](x), \qquad (29)$$

which implies that

$$v[t](x) = \int_0^x k(x,y)u[t](y)\mathrm{d}y - \int_0^x k(x,y)v[t](y)\mathrm{d}y. \quad (30)$$

For any fixed $t \in \mathbb{R}_{>0}$, we prove the unique existence of a continuous solution $v[t]$ to the integral equation (30).

Define

$$v_0[t](x) := \int_0^x k(x,y)u[t](y)\mathrm{d}y$$

and

$$v_n[t](x) := -\int_0^x k(x,y)v_{n-1}[t](y)\mathrm{d}y, \forall n \in \mathbb{N}_+.$$

It follows that

$$|v_0[t](x)| \leq \alpha_1\|u[t]\|_{L^p(0,1)},$$

and, by induction, that

$$|v_n[t](x)| \leq \frac{\alpha_1^{n+1}}{n!}\|u[t]\|_{L^p(0,1)}x^n, \forall n \in \mathbb{N}_+. \qquad (31)$$

Then, for any fixed $t \in \mathbb{R}_{>0}$, the series $\sum_{n=0}^\infty v_n[t](x)$ converges absolutely and uniformly in $x \in [0,1]$, and its sum $v[t](x) := \sum_{n=0}^\infty v_n[t](x)$ is a continuous solution to (30). Moreover, (31) implies that

$$|v[t](x)| \leq \sum_{n=0}^\infty |v_n[t](x)|$$
$$\leq \sum_{n=0}^\infty \frac{\alpha_1^{n+1}}{n!}\|u[t]\|_{L^p(0,1)}x^n$$
$$= \alpha_1\|u[t]\|_{L^p(0,1)}e^{\alpha_1 x}.$$

Therefore, there exists a positive constant $M_1 := \alpha_1^{1-\frac{1}{p}}\left(\frac{e^{p\alpha_1}-1}{p}\right)^{\frac{1}{p}}$ for $p \in [1,+\infty)$, or $M_1 := \alpha_1 e^{\alpha_1}$ for $p = +\infty$, such that

$$\|v[t]\|_{L^p(0,1)} \leq M_1\|u[t]\|_{L^p(0,1)}. \qquad (32)$$

It follows from (30) and (32) that, for any fixed $t \in \mathbb{R}_{>0}$, there exists a bounded linear operator $\Psi: L^p(0,1) \to L^p(0,1)$ such that $v[t](x) = (\Psi u)[t](x)$.

By (29) and (32), we get

$$w[t](x) = ((I - \Psi)u)[t](x) = (K^{-1}u)[t](x),$$

and

$$\|w[t]\|_{L^p(0,1)} \leq (1 + M_1)\|u[t]\|_{L^p(0,1)}, \qquad (33)$$

which implies that $K^{-1}: L^p(0,1) \to L^p(0,1)$ is uniformly bounded with respect to t.

Step 2. For $\mathscr{D} = W^{1,p}(0,1)$, we prove that K has a uniformly bounded inverse $K^{-1}: W^{1,p}(0,1) \to W^{1,p}(0,1)$.

Indeed, the estimate $|v_0[t](x)| \leq \alpha_1\|u[t]\|_{L^p(0,1)}$ and (31) imply that

$$|v_n[t](x)| \leq \frac{\alpha_1^{n+1}}{n!}\|u[t]\|_{L^p(0,1)}x^n, \forall n \in \mathbb{N}_0. \qquad (34)$$

By the definition of v_n, for any fixed t, we have

$$\frac{\partial v_n}{\partial x}[t](x) = -k(x,x)v_{n-1}[t](x)$$
$$- \int_0^x k_x(x,y)v_{n-1}[t](y)\mathrm{d}y, \forall n \in \mathbb{N}_+. \quad (35)$$

It follows from the definition of v_0, (34) and (35) that

$$\left|\frac{\partial v_0}{\partial x}[t](x)\right| \leq (\alpha_2 + \alpha_3)\|u[t]\|_{L^p(0,1)}, \qquad (36a)$$

$$\left|\frac{\partial v_n}{\partial x}[t](x)\right| \leq \alpha_2|v_{n-1}[t](x)| + \alpha_3\int_0^x |v_{n-1}[t](y)|\mathrm{d}y$$

$$\leq \frac{(\alpha_2+\alpha_3)\alpha_1^n}{(n-1)!}\|u[t]\|_{L^p(0,1)}x^{n-1}, \forall n \in \mathbb{N}_+,$$
(36b)

which implies that $\sum_{n=0}^{\infty}\frac{\partial v_n}{\partial x}[t](x)$ uniformly converges in $[0,1]$ for any fixed t. Therefore, for any fixed t, $\frac{\partial v}{\partial x}[t](x)$ exists in $[0,1]$, and $\frac{\partial v}{\partial x}[t](x) = \sum_{n=0}^{\infty}\frac{\partial v_n}{\partial x}[t](x)$, which belongs to $C([0,1])$.

By (30), it follows that

$$\frac{\partial v}{\partial x}[t](x) = k(x,x)(u[t](x)-v[t](x))$$
$$+ \int_0^x k_x(x,y)(u[t](y)-v[t](y))\mathrm{d}y.$$

By the Cauchy-Schwarz inequality, it holds that

$$\left\|\frac{\partial v}{\partial x}[t]\right\|_{L^p(0,1)}^p \leq M_2^p\|u[t]\|_{L^p(0,1)}^p,$$
(37)

where $M_2 := 4(1+M_1)(\alpha_2+\alpha_3)$.

By (32) and (37), for any fixed t, we obtain

$$\|w[t]\|_{W^{1,p}(0,1)} = \|u[t]-v[t]\|_{W^{1,p}(0,1)}$$
$$\leq M_3\|u[t]\|_{W^{1,p}(0,1)},$$
(38)

where $M_3 := 1+M_1+M_2$ for $p \in [1,+\infty)$, or $M_3 := 1+\alpha_1 e^{\alpha_1}+4(1+\alpha_1 e^{\alpha_1})(\alpha_2+\alpha_3)$ for $p = +\infty$. Thus $K^{-1}: W^{1,p}(0,1) \to W^{1,p}(0,1)$ is uniformly bounded with respect to t. ∎

Proof of Lemma 3: We prove the result in a similar way as for Lemma 2.

We prove the unique existence of a continuous solution $v(x,t)$ to the integral equation

$$v(x,t) = \int_0^x k(x,y)u(y,t)\mathrm{d}y - \int_0^x k(x,y)v(y,t)\mathrm{d}y. \quad (39)$$

Let v_n with $n \in \mathbb{N}_0$ be defined as in the proof of Lemma 2, i.e.,

$$v_0(x,t) := \int_0^x k(x,y)u(y,t)\mathrm{d}y$$

and

$$v_n(x,t) := -\int_0^x k(x,y)v_{n-1}(y,t)\mathrm{d}y, \forall n \in \mathbb{N}_+.$$

By the definition of v_n, we have

$$\frac{\partial^2 v_n}{\partial x^2}(x,t) = \left(-\frac{\mathrm{d}}{\mathrm{d}x}(k(x,x)) - (k_x(x,y))|_{y=x}\right)v_{n-1}(x,t)$$
$$- \int_0^x k_{xx}(x,y)v_{n-1}(y,t)\mathrm{d}y$$
$$- k(x,x)\frac{\partial v_{n-1}}{\partial x}(x,t), \forall n \in \mathbb{N}_+. \quad (40)$$

In view of (34), (36) and (40), it holds that for any $(x,t) \in \overline{Q}_T$:

$$|v_n(x,t)| \leq \frac{\alpha_1^{n+1}}{n!}\|u(\cdot,t)\|_{L^p(0,1)}x^n$$
$$\leq \frac{\alpha_1^{n+1}}{n!}\|u\|_{C^{2,1}(\overline{Q}_T)}x^n, \forall n \in \mathbb{N}_0, \quad (41a)$$

$$\left|\frac{\partial v_0}{\partial x}(x,t)\right| \leq (\alpha_2+\alpha_3)\|u(\cdot,t)\|_{L^p(0,1)}$$
$$\leq (\alpha_2+\alpha_3)\|u\|_{C^{2,1}(\overline{Q}_T)}, \quad (41b)$$

$$\left|\frac{\partial v_n}{\partial x}(x,t)\right| \leq \frac{(\alpha_2+\alpha_3)\alpha_1^n}{(n-1)!}\|u(\cdot,t)\|_{L^p(0,1)}x^{n-1}$$
$$\leq \frac{(\alpha_2+\alpha_3)\alpha_1^n}{(n-1)!}\|u\|_{C^{2,1}(\overline{Q}_T)}x^{n-1}, \forall n \in \mathbb{N}_+, \quad (41c)$$

$$\left|\frac{\partial^2 v_0}{\partial x^2}(x,t)\right| \leq (\alpha_2+\alpha_4+\alpha_5)\|u(\cdot,t)\|_{L^p(0,1)}$$
$$\leq (\alpha_2+\alpha_4+\alpha_5)\|u\|_{C^{2,1}(\overline{Q}_T)}, \quad (41d)$$

$$\left|\frac{\partial^2 v_1}{\partial x^2}(x,t)\right| \leq \alpha_6\|u(\cdot,t)\|_{L^p(0,1)}$$
$$\leq \alpha_6\|u\|_{C^{2,1}(\overline{Q}_T)}, \quad (41e)$$

$$\left|\frac{\partial^2 v_n}{\partial x^2}(x,t)\right| \leq \alpha_5|v_{n-1}(x,t)| + \alpha_2\left|\frac{\partial v_{n-1}}{\partial x}(x,t)\right|$$
$$+ \int_0^x \alpha_4|v_{n-1}(y,t)|\mathrm{d}y$$
$$\leq \alpha_6\frac{\alpha_1^{n-1}}{(n-2)!}\|u(\cdot,t)\|_{L^p(0,1)}x^{n-2}$$
$$\leq \alpha_6\frac{\alpha_1^{n-1}}{(n-2)!}\|u\|_{C^{2,1}(\overline{Q}_T)}x^{n-2}, \forall n \geq 2, \quad (41f)$$

where $\alpha_6 := \alpha_1\alpha_5+\alpha_2^2+\alpha_2\alpha_3+\alpha_1\alpha_4$.

Analogous to (31), for any $(x,t) \in \overline{Q}_T$, we have

$$\left|\frac{\partial v_n}{\partial t}(x,t)\right| \leq \frac{\alpha_1^{n+1}}{n!}\left\|\frac{\partial u}{\partial t}(\cdot,t)\right\|_{L^p(0,1)}x^n$$
$$\leq \frac{\alpha_1^{n+1}}{n!}\|u\|_{C^{2,1}(\overline{Q}_T)}x^n, \forall n \in \mathbb{N}_0. \quad (42)$$

By (41) and (42), we deduce that
- $\sum_{n=0}^{\infty}v_n(x,t)$ uniformly converge in $\overline{Q}_T$;
- $\sum_{n=0}^{\infty}\frac{\partial v_n}{\partial x}(x,t)$ uniformly converge in $\overline{Q}_T$;
- $\sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial x^2}(x,t)$ uniformly converge in $\overline{Q}_T$;
- $\sum_{n=0}^{\infty}\frac{\partial v_n}{\partial t}(x,t)$ uniformly converge in $\overline{Q}_T$.

Then $v(x,t) := \sum_{n=0}^{\infty}v_n(x,t)$ is a continuous solution to (39), and satisfies
- $\frac{\partial v}{\partial x}(x,t) = \sum_{n=0}^{\infty}\frac{\partial v_n}{\partial x}(x,t)$ that belongs to $C(\overline{Q}_T)$;
- $\frac{\partial^2 v}{\partial x^2}(x,t) = \sum_{n=0}^{\infty}\frac{\partial^2 v_n}{\partial x^2}(x,t)$ that belongs to $C(\overline{Q}_T)$;
- $\frac{\partial v}{\partial t}(x,t) = \sum_{n=0}^{\infty}\frac{\partial v_n}{\partial t}(x,t)$ that belongs to $C(\overline{Q}_T)$.

Furthermore, we infer from (41) and (42) that there is a positive constant M_4 such that

$$\|v\|_{C^{2,1}(\overline{Q}_T)} \leq M_4\|u\|_{C^{2,1}(\overline{Q}_T)}.$$

We conclude that K has a bounded linear inverse $K^{-1}: C^{2,1}(\overline{Q}_T) \to C^{2,1}(\overline{Q}_T)$. ∎

We are ready for the proof of the main result obtained in this paper.

Proof of Theorem 1: The existence, uniqueness and regularity of a solution to the closed-loop system (1) are guaranteed by Lemma 1 and Lemma 3.

Now we show that the closed-loop system (1) is L^r-ISS both in the spatial L^p-norm and in the spatial $W^{1,p}$-norm.

Indeed, by the definition of u_0 and the Cauchy-Schwarz inequality, we have

$$\|u_0\|_{L^p(0,1)} \leq 2(1+\alpha_1)\|w_0\|_{L^p(0,1)}, \tag{43a}$$

$$\|u_{0x}\|_{L^p(0,1)} \leq 3(1+\alpha_2+\alpha_3)\|w_0\|_{W^{1,p}(0,1)}. \tag{43b}$$

For $p \in [1,+\infty]$ and $r \in [p,+\infty]$, applying Proposition 1, Lemma 2 with $\mathscr{D} = L^p(0,1)$ and $\mathscr{D} = W^{1,p}(0,1)$, respectively (see (33) and (38)), and using (43), we obtain for any $t \in \mathbb{R}_{>0}$:

$$\|w(\cdot,t)\|_{L^p(0,1)} \leq C \left(e^{-\underline{\lambda}t}\|w_0\|_{L^p(0,1)} + R\|f\|_{L^r((0,t);L^p(0,1))}\right), \tag{44}$$

and

$$\|w(\cdot,t)\|_{W^{1,p}(0,1)} \leq C \left(e^{-\underline{\lambda}t}\|w_0\|_{W^{1,p}(0,1)} + R\|f\|_{L^r((0,t);W^{1,p}(0,1))}\right), \tag{45}$$

where $C > 0$ is a constant depending only on $\alpha_1, \alpha_2, \alpha_3, M_1$ and M_3, R and $\underline{\lambda}$ are defined in Proposition 1, and M_1 and M_3 are defined in the proof of Lemma 2.

The estimates (44) and (45) indicate that system (1) is L^r-ISS in the spatial L^p-norm and $W^{1,p}$-norm, respectively. ∎

V. Conclusion

In this paper, the integral input-to-state stabilization in different norms was studied for certain linear parabolic PDEs with external inputs. A boundary backstepping controller was designed, and the iISS of the closed-loop system was analyzed by employing the ALM that can avoid singularities in the CLM.

As applying the backstepping to nonlinear PDE systems is more challenging, our future work will focus on introducing novel techniques for the control design and address the integral input-to-state stabilization in different norms for nonlinear PDE systems with external inputs.

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